
A dynamic programming approach to finite-horizon coherent quantum LQG control

Igor G. Vladimirov · Ian R. Petersen

Abstract The paper is concerned with the coherent quantum Linear Quadratic Gaussian (CQLQG) control problem for time-varying quantum plants governed by linear quantum stochastic differential equations over a bounded time interval. A controller is sought among quantum linear systems satisfying physical realizability (PR) conditions. The latter describe the dynamic equivalence of the system to an open quantum harmonic oscillator and relate its state-space matrices to the free Hamiltonian, coupling and scattering operators of the oscillator. Using the Hamiltonian parameterization of PR controllers, the CQLQG problem is recast into an optimal control problem for a deterministic system governed by a differential Lyapunov equation. The state of this subsidiary system is the symmetric part of the quantum covariance matrix of the plant-controller state vector. The resulting covariance control problem is treated using dynamic programming and Pontryagin's minimum principle. The associated Hamilton-Jacobi-Bellman equation for the minimum cost function involves Frechet differentiation with respect to matrix-valued variables. The gain matrices of the CQLQG optimal controller are shown to satisfy a quasi-separation property as a weaker quantum counterpart of the filtering/control decomposition of classical LQG controllers.

Keywords Quantum control · LQG cost · Physical realizability · Symplectic invariance · Dynamic programming · Pontryagin minimum principle · Frechet differentiation

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1 Introduction

Quantum feedback control, which deals with dynamical systems whose variables are non-commutative linear operators on a Hilbert space governed by the laws of quantum mechanics (see, for example, [24]), involves two major paradigms. One of them employs classical information on the quantum mechanical system retrieved through a macroscopic measuring device and thus accompanied by decoherence and the loss of quantum information, as reflected, for example, in the projection postulate [6]. The other, less “invasive”, approach, apparently practiced by nature to stabilize matter on an atomic scale, is through direct interaction of quantum mechanical systems, possibly mediated by light fields. With the advances in quantum optics and nanotechnology making it possible to manipulate such interconnection, measurement-free coherent quantum controllers provide an important alternative to the classical observation-actuation control loop.

Coherent quantum feedback can be implemented, for example, using quantum-optical components, such as optical cavities, beam splitters, phase shifters, and modelled by linear quantum stochastic differential equations (QSDEs) [7, 11, 13] corresponding to open quantum harmonic oscillators [2, 4]. The associated notion of physical realizability (PR) [8, 14, 16] reflects the dynamic equivalence of a system to an open quantum harmonic oscillator. Being organized as quadratic constraints on the state-space matrices, the PR conditions imposed on the controller complicate the solution of quantum analogues to the classical Linear Quadratic Gaussian (LQG) and \mathcal{H}_∞ -control problems, and it is particularly so in regard to the Coherent Quantum LQG (CQLQG) problem [12] which has yet to be solved.

The CQLQG control problem seeks a PR quantum controller to minimize the average output “energy” of the closed-loop system, described by the quantum expectation of a quadratic form of its state variables. For the original infinite-horizon time-invariant setting of the problem, a numerical procedure was proposed in [12] to compute *suboptimal* controllers, and algebraic equations were obtained in [22] for the *optimal* CQLQG controller. Subtle coupling of the equations, which comes from the PR constraints, is apparently related to the complicated nature of the suboptimal control design procedure. This suggests that an alternative viewpoint needs to be taken for a better understanding of the structure of the optimal quantum controller.

In the present paper, the CQLQG control problem is approached by considering its time-varying version, with a PR quantum controller being sought to minimize the average output energy of the closed-loop system over a bounded time interval. We outline a dynamic programming approach to the finite-horizon time-varying CQLQG problem by recasting it as a deterministic optimal control problem for a dynamical system governed by a differential Lyapunov equation. The state of the subsidiary system is the symmetric part of the quantum covariance matrix of the plant-controller state vector. The role of control in this covariance control [18] problem is played by a triple of matrices from the Hamiltonian parameterization of a PR controller which relates its state-space matrices to the free Hamiltonian, coupling and scattering operators of an open quantum harmonic oscillator [2].

The dynamic programming approach to the covariance control problem is developed in conjunction with Pontryagin’s minimum principle [15]. The appropriate costate of the subsidiary dynamical system is shown to coincide with the observability Gramian of the underlying closed-loop system. The resulting Hamilton-Jacobi-Bellman equation (HJBE) for the minimum cost function of the closed-loop system state covariance matrix involves Frechet differentiation in noncommutative matrix-valued variables. Such partial differential equations (PDEs) were considered, for example, in [20] in a different context of entropy variational problems for Gaussian diffusion processes.

Using the invariance of PR quantum controllers under the group of symplectic equivalence transformations of the state-space matrices (which is a salient feature of such controllers), we establish *symplectic invariance* of the minimum cost function. This reduces the minimization of Pontryagin's control Hamiltonian [19] to two independent quadratic optimization problems which yield the gain matrices of the optimal CQLQG controller. As in the time-invariant case [22], this partial decoupling is a weaker quantum counterpart of the filtering/control separation principle of classical LQG controllers [9]. The equations for the optimal quantum controller involve the inverse of special self-adjoint operators on matrices [22], which can be carried out through the matrix vectorization [10, 18].

The paper is organised as follows. Sections 2 and 3 specify the quantum plants and coherent quantum controllers being considered. Section 4 revisits PR conditions to make the exposition self-contained. Section 5 formulates the CQLQG control problem. Section 6 derives a PDE for the minimum cost function from the symplectic invariance of PR controllers. This PDE is solved in Appendix A to provide an insight into the structure of the function. Section 7 establishes the HJBE for the minimum cost function and identifies the costate in the related Pontryagin's minimum principle as the observability Gramian. Section 8 carries out the minimization involved in the HJBE and obtains equations for the optimal controller gain matrices, using the special linear operators from Appendix B. Section 9 summarizes the system of equations for the optimal CQLQG controller. Section 10 provides concluding remarks and outlines further research.

2 Open quantum plant

The quantum plant considered below is an open quantum system which is coupled to another such system (playing the role of a controller), with the dynamics of both systems affected by the environment. At any time t , the plant is described by a n -dimensional vector x_t of self-adjoint operators on a Hilbert space, with n even. The plant state vector x_t evolves in time and contributes to a p_1 -dimensional output of the plant y_t (also with self-adjoint operator-valued entries) according to QSDEs

$$dx_t = A_t x_t dt + B_t dw_t + E_t d\eta_t, \quad (1)$$

$$dy_t = C_t x_t dt + D_t dw_t. \quad (2)$$

Here, the matrices $A_t \in \mathbb{R}^{n \times n}$, $B_t \in \mathbb{R}^{n \times m_1}$, $C_t \in \mathbb{R}^{p_1 \times n}$, $D_t \in \mathbb{R}^{p_1 \times m_1}$, $E_t \in \mathbb{R}^{n \times p_2}$ are known deterministic functions of time, which are assumed to be continuous for well-posedness of the QSDEs,

$$z_t := C_t x_t \quad (3)$$

is the “signal part” of the plant output y_t , and η_t is the output of the controller to be described in Section 3. The noise from the environment is represented by an m_1 -dimensional quantum Wiener process w_t (with m_1 even) on the boson Fock space [13] with a canonical Ito table

$$dw_t dw_t^T = (I_{m_1} + iJ_1/2)dt. \quad (4)$$

Here, i is the imaginary unit, I_m is the identity matrix of order m (with the subscript sometimes omitted), and J_1 is a real antisymmetric matrix, which is given by

$$J_1 := I_{\mu_1} \otimes \mathbf{J}, \quad \mathbf{J} := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad (5)$$

(with \otimes the Kronecker product of matrices, and $\mu_1 := m_1/2$) and specifies the canonical commutation relations (CCRs) for the quantum noise of the plant as

$$[dw_t, dw_t^T] := dw_t dw_t^T - (dw_t dw_t^T)^T = iJ_1 dt. \quad (6)$$

Vectors are assumed to be organized as columns unless indicated otherwise, and the transpose $(\cdot)^T$ acts on vectors and matrices with operator-valued entries as if the latter were scalars. Accordingly, the (j, k) th entry of the matrix $[W, W^T]$, associated with a vector W of operators W_1, \dots, W_r , is the commutator

$$[W_j, W_k] := W_j W_k - W_k W_j.$$

Also, $(\cdot)^\dagger := ((\cdot)^\#)^T$ denotes the transpose of the entry-wise adjoint $(\cdot)^\#$. In application to ordinary matrices, $(\cdot)^\dagger$ is the complex conjugate transpose $(\overline{(\cdot)})^T$ and will be written as $(\cdot)^*$.

3 Coherent quantum controller

A measurement-free coherent quantum controller is another quantum system with a n -dimensional state vector ξ_t with self-adjoint operator-valued entries whose interconnection with the plant (1)–(3) is described by the QSDEs

$$d\xi_t = a_t \xi_t dt + b_t d\omega_t + e_t dy_t, \quad (7)$$

$$d\eta_t = c_t \xi_t dt + d_t d\omega_t. \quad (8)$$

Here, $a_t \in \mathbb{R}^{n \times n}$, $b_t \in \mathbb{R}^{n \times m_2}$, $c_t \in \mathbb{R}^{p_2 \times n}$, $d_t \in \mathbb{R}^{p_2 \times m_2}$, $e_t \in \mathbb{R}^{n \times p_1}$ are deterministic continuous functions of time, and, similarly to (3), the process

$$\zeta_t := c_t \xi_t \quad (9)$$

is the signal part of the controller output η_t . The process ω_t in (7) and (8) is the controller noise which is assumed to be an m_2 -dimensional quantum Wiener process (with m_2 even) that commutes with the plant noise w_t in (1) and (2) and also has a canonical Ito table $d\omega_t d\omega_t^T = (I_{m_2} + iJ_2/2)dt$ with the CCR matrix

$$J_2 := I_{\mu_2} \otimes \mathbf{J}, \quad (10)$$

where $\mu_2 := m_2/2$. In view of (7), the matrices b_t and e_t will be referred to as the controller noise and observation gain matrices, even though y_t is not an observation signal in the classical control theoretic sense. The combined set of equations (1)–(3) and (7)–(9) describes the fully quantum closed-loop system shown in Fig. 1. The process ζ_t in (9) is analogous to the

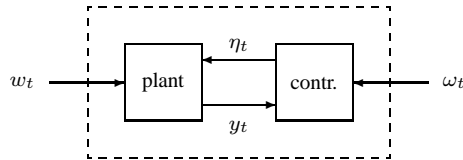


Fig. 1 The plant and controller form a closed-loop quantum system described by (1)–(3) and (7)–(9), which is influenced by the environment through the quantum Wiener processes w_t and ω_t .

actuator signal in classical control theory. Following the classical approach, the performance of the quantum controller is described in terms of an r -dimensional process

$$\mathcal{Z}_t = F_t x_t + G_t \zeta_t. \quad (11)$$

Its entries are linear combinations of the plant state and “actuator output” variables whose relative importance is specified by the matrices $F_t \in \mathbb{R}^{r \times n}$ and $G_t \in \mathbb{R}^{r \times m_2}$ which are known continuous deterministic functions of time t . The weighting matrices F_t, G_t are free from physical constraints and their choice is dictated by the control design preferences. The $2n$ -dimensional combined state vector

$$\mathcal{X}_t := \begin{bmatrix} x_t \\ \xi_t \end{bmatrix}$$

of the closed-loop system and the “output” \mathcal{Z}_t in (11) are governed by the QSDEs

$$d\mathcal{X}_t = \mathcal{A}_t \mathcal{X}_t dt + \mathcal{B}_t d\mathcal{W}_t, \quad \mathcal{Z}_t = \mathcal{C}_t \mathcal{X}_t, \quad (12)$$

driven by the combined $(m_1 + m_2)$ -dimensional quantum Wiener process

$$\mathcal{W}_t := \begin{bmatrix} w_t \\ \omega_t \end{bmatrix}$$

with a block diagonal Ito table

$$d\mathcal{W}_t d\mathcal{W}_t^T = (I_{m_1+m_2} + iJ/2)dt, \quad J := I_{\mu_1+\mu_2} \otimes \mathbf{J} \quad (13)$$

in conformance with (4)–(6), (10). The state-space matrices of the closed-loop system (12) are given by

$$\left[\begin{array}{c|c} \mathcal{A}_t & \mathcal{B}_t \\ \hline \mathcal{C}_t & 0 \end{array} \right] = \left[\begin{array}{cc|cc} A_t & E_t c_t & B_t & E_t d_t \\ e_t C_t & a_t & e_t D_t & b_t \\ \hline F_t & G_t c_t & 0 & 0 \end{array} \right]. \quad (14)$$

4 Physical realizability conditions

The CCRs for the closed-loop system state vector are described by a real antisymmetric matrix

$$\Theta_t := -i[\mathcal{X}_t, \mathcal{X}_t^T] = 2\text{Im}\mathbf{E}(\mathcal{X}_t \mathcal{X}_t^T), \quad (15)$$

which, up to a factor of 2, coincides with the entrywise imaginary part of the quantum covariance matrix of \mathcal{X}_t , with $\mathbf{E}(\cdot)$ the quantum expectation (associated, in what follows, with the vacuum state). The matrix Θ_t evolves in time according to a differential Lyapunov equation

$$\dot{\Theta}_t = \mathcal{A}_t \Theta_t + \Theta_t \mathcal{A}_t^T + \mathcal{B}_t J \mathcal{B}_t^T, \quad (16)$$

where J is the CCR matrix of the combined quantum Wiener process \mathcal{W}_t from (13) in the sense that $[d\mathcal{W}_t, d\mathcal{W}_t^T] = iJdt$. Indeed, by employing the ideas of [8, Proof of Theorem 2.1 on pp. 1798–1799] and combining the quantum Ito formula with the bilinearity of the commutator as

$$d[X, Y] = [dX, Y] + [X, dY] + [dX, dY],$$

and using the adaptedness of the state process \mathcal{X}_t and the quantum Ito product rules [13], it follows that

$$\begin{aligned} d[\mathcal{X}_t, \mathcal{X}_t^T] &= [\mathcal{A}_t \mathcal{X}_t dt + \mathcal{B}_t d\mathcal{W}_t, \mathcal{X}_t^T] + [\mathcal{X}_t, \mathcal{X}_t^T \mathcal{A}_t^T dt + d\mathcal{W}_t^T \mathcal{B}_t^T] \\ &\quad + [\mathcal{A}_t \mathcal{X}_t dt + \mathcal{B}_t d\mathcal{W}_t, \mathcal{X}_t^T \mathcal{A}_t^T dt + d\mathcal{W}_t^T \mathcal{B}_t^T] \\ &= (\mathcal{A}_t [\mathcal{X}_t, \mathcal{X}_t^T] + [\mathcal{X}_t, \mathcal{X}_t^T] \mathcal{A}_t^T) dt + \mathcal{B}_t [d\mathcal{W}_t, d\mathcal{W}_t^T] \mathcal{B}_t^T, \end{aligned}$$

which, upon division by i , yields (16). Now, suppose the initial plant and controller state vectors commute with each other, so that $[x_0, \xi_0^T] = 0$ and the matrix (15) is initialized by

$$\Theta_0 = \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix}, \quad (17)$$

where K_1, K_2 are nonsingular real antisymmetric matrices. Then the CCR matrix of \mathcal{X}_t is preserved in time if and only if

$$\mathcal{A}_t \Theta_0 + \Theta_0 \mathcal{A}_t^T + \mathcal{B}_t J \mathcal{B}_t^T = 0 \quad (18)$$

for any $t \geq 0$. The left-hand side of (18) is always an antisymmetric matrix. Hence, by computing two diagonal and one off-diagonal blocks of this matrix with the aid of (14), it follows that the CCR preservation is equivalent to three equations

$$A_t K_1 + K_1 A_t^T + B_t J_1 B_t^T + E_t d_t J_2 d_t^T E_t^T = 0, \quad (19)$$

$$E_t \underbrace{(c_t K_2 + d_t J_2 b_t^T)}_{\text{controller}} + \underbrace{(K_1 C_t^T + B_t J_1 D_t^T)}_{\text{plant}} e_t^T = 0, \quad (20)$$

$$a_t K_2 + K_2 a_t^T + e_t D_t J_1 D_t^T e_t^T + b_t J_2 b_t^T = 0 \quad (21)$$

to be satisfied at any time t . Therefore, the fulfillment of the equalities

$$c_t K_2 + d_t J_2 b_t^T = 0, \quad (22)$$

$$C_t K_1 + D_t J_1 B_t^T = 0, \quad (23)$$

which pertain to the controller and the plant, respectively, is sufficient for (20). Note that (21) and (22) are the conditions of physical realizability (PR) [8, 12] of the quantum controller in the sense of equivalence of its input-output operator to an open quantum harmonic oscillator [4]. In a similar fashion, (19) and (23) are the PR conditions for the quantum plant.

Lemma 1 *Suppose the quantum plant satisfies the PR conditions (19), (23) and the matrix E_t is of full column rank. Then the closed-loop system state CCR matrix Θ_0 in (17) is preserved if and only if the controller satisfies the PR conditions (21), (22).*

Proof The “if” part of the lemma was considered above. The “only if” claim is established by left multiplying both sides of (20) by $(E_t^T E_t)^{-1} E_t^T$. This is valid if E_t is of full column rank and yields

$$c_t K_2 + d_t J_2 b_t^T = -(E_t^T E_t)^{-1} E_t^T (K_1 C_t^T + B_t J_1 D_t^T).$$

In this case, the fulfillment of the second PR condition for the plant (23) indeed entails the PR condition (22) for the controller. ■

The fact that the CCR preservation property for the closed-loop system state alone “covers” the separate PR conditions for the plant and controller as input-output operators is explained by the “internalization” of their outputs which become part of the state dynamics when the systems are coupled. In what follows, the controller state CCR matrix is assumed to be given by

$$K_2 = I_\nu \otimes \mathbf{J} =: J_0, \quad (24)$$

where $\nu := n/2$, so that the controller PR conditions (21) and (22) take the form

$$a_t J_0 + J_0 a_t^T + e_t D_t J_1 D_t^T e_t^T + b_t J_2 b_t^T = 0, \quad (25)$$

$$c_t J_0 + d_t J_2 b_t^T = 0. \quad (26)$$

The first PR condition (25) is a linear equation with respect to a_t whose solutions are parameterized by real symmetric matrices R_t of order n as

$$a_t = \underbrace{(e_t D_t J_1 D_t^T e_t^T + b_t J_2 b_t^T) J_0 / 2}_{\tilde{a}_t} + J_0 R_t. \quad (27)$$

These solutions form an affine subspace in $\mathbb{R}^{n \times n}$ obtained by translating the linear subspace of Hamiltonian matrices

$$\{a \in \mathbb{R}^{n \times n} : a J_0 + J_0 a^T = 0\} = J_0 \mathbb{S}_n = \mathbb{S}_n J_0$$

by a skew-Hamiltonian matrix \tilde{a}_t , that is, a particular solution of (25) which is a quadratic function of b_t and e_t . Here, \mathbb{S}_n denotes the subspace of real symmetric matrices of order n , and $R_t \in \mathbb{S}_n$ specifies the free Hamiltonian operator $\xi_t^T R_t \xi_t / 2$ of the quantum harmonic oscillator [2, Eqs. (20)–(22) on pp. 8–9]. Note that (27) is the orthogonal decomposition of a_t into projections onto the subspaces of skew-Hamiltonian and Hamiltonian matrices in the sense of the Frobenius inner product

$$\langle X, Y \rangle := \text{Tr}(X^T Y).$$

Since the canonical structure of J_0 in (24) implies that $J_0^{-1} = -J_0$, the second PR condition (26) allows the matrix c_t to be expressed in terms of b_t as

$$c_t = d_t J_2 b_t^T J_0. \quad (28)$$

The matrix d_t , which quantifies the instantaneous gain of the controller output η_t with respect to the controller noise ω_t , is assumed to be fixed. Then (27), (28) completely parameterize the state-space matrices of a PR controller by the matrix triple

$$u_t := (b_t, e_t, R_t), \quad (29)$$

which will be regarded as an element of the Hilbert space $\mathbb{U} := \mathbb{R}^{n \times m_2} \times \mathbb{R}^{n \times p_1} \times \mathbb{S}_n$ with the inherited inner product $\langle (b, e, R), (\beta, \epsilon, \rho) \rangle := \langle b, \beta \rangle + \langle e, \epsilon \rangle + \langle R, \rho \rangle$.

5 Coherent quantum LQG control problem

In extending the infinite-horizon time invariant case from [12, 22], the CQLQG control problem is formulated as the minimization of the average output “energy” of the closed-loop system (12) over a bounded time interval $[0, T]$:

$$\mathcal{E}_T := \int_0^T \mathbf{E}(Z_t^T Z_t) dt = \int_0^T \langle \mathcal{C}_t^T \mathcal{C}_t, P_t \rangle dt \longrightarrow \min, \quad (30)$$

where the minimum is taken over the maps $t \mapsto u_t$ in (29) which parameterize the n -dimensional controllers (7)–(9) satisfying the PR conditions (25), (26). Here, $Z_t^T Z_t$ is the sum of squared entries of the vector Z_t from (11), which are self-adjoint quantum mechanical operators. Also,

$$P_t := \text{Re} \mathbf{E}(\mathcal{X}_t \mathcal{X}_t^T) \quad (31)$$

is a real positive semi-definite symmetric matrix of order $2n$ (we denote the set of such matrices by \mathbb{S}_{2n}^+) which is the entrywise real part of the quantum covariance matrix of the closed-loop system state vector \mathcal{X}_t . The matrix P_t satisfies the differential Lyapunov equation

$$\dot{P}_t = \mathcal{A}_t P_t + P_t \mathcal{A}_t^T + \mathcal{B}_t \mathcal{B}_t^T =: \mathcal{L}_{t, u_t}(P_t). \quad (32)$$

The affine operator \mathcal{L}_{t, u_t} is the infinitesimal generator of a two-parameter semi-group, which acts on \mathbb{S}_{2n}^+ and depends on the triple u_t of current matrices of the PR controller from (29) through (14), (27), (28). If P_0 were zero (which is forbidden by the positive semi-definiteness of the quantum covariance matrix $\mathbf{E}(\mathcal{X}_0 \mathcal{X}_0^T) = P_0 + i\theta_0/2 \succcurlyeq 0$), then P_t would coincide with the controllability Gramian, over the time interval $[0, t]$, of a classical linear time-varying system with the state-space realization triple $(\mathcal{A}_t, \mathcal{B}_t, \mathcal{C}_t)$ driven by a standard Wiener process. The fact that \mathcal{E}_T in (30) is representable as the LQG cost of a classical system reduces the CQLQG problem to a constrained LQG control problem for an equivalent classical plant

$$\left[\begin{array}{c|cc} A_t & B_t & E_t dt \\ \hline F_t & 0 & 0 \\ \hline 0 & 0 & I \\ \hline C_t & D_t & 0 \end{array} \right] \quad (33)$$

driven by an $(m_1 + m_2)$ -dimensional standard Wiener process, with the controller being noiseless. In accordance with the standard convention, the block structure of the state-space realization in (33) corresponds to partitioning the input into the noise and control, and the output into the to-be-controlled and observation signals. We will develop a dynamic programming approach to (30) as an optimal control problem for a subsidiary dynamical system with state P_t in (31) governed by the ODE (32) whose right-hand side is specified by the matrix triple u_t from (29). With the time horizon T assumed to be fixed, the minimum cost function is defined by

$$V_t(P) := \inf \int_t^T \langle \mathcal{C}_s^T \mathcal{C}_s, P_s \rangle ds \quad (34)$$

for any $t \in [0, T]$ and $P \in \mathbb{S}_{2n}^+$. Here, the infimum is taken over all admissible state-space matrices of the PR controller on the time interval $[t, T]$, provided the initial symmetric covariance matrix of the closed-loop system state vector is $\text{Re} \mathbf{E}(\mathcal{X}_t \mathcal{X}_t^T) = P$.

6 Symplectic invariance

As in the time-invariant case [22], the PR conditions (25)–(26) are invariant with respect to the group of *symplectic* similarity transformations of the controller matrices

$$a_t \mapsto \sigma a_t \sigma^{-1}, \quad b_t \mapsto \sigma b_t, \quad e_t \mapsto \sigma e_t, \quad c_t \mapsto c_t \sigma^{-1},$$

where σ is an arbitrary (possibly, time-varying) symplectic matrix of order n in the sense that $\sigma J_0 \sigma^T = J_0$. This corresponds to the canonical state transformation $\xi_t \mapsto \sigma \xi_t$; see also [17, Eqs. (12)–(14)]. Any such transformation of a PR controller leads to its equivalent state-space representation, with the matrix R_t transformed as $R_t \mapsto \sigma^{-T} R_t \sigma^{-1}$, where $(\cdot)^{-T} := ((\cdot)^{-1})^T$. Hence, the minimum cost function $V_t(P)$ in (34) is invariant under the corresponding group of transformations of the closed-loop system state covariance matrix P , that is,

$$V_t(SPS^T) = V_t(P), \quad S := \begin{bmatrix} I & 0 \\ 0 & \sigma \end{bmatrix} \quad (35)$$

for any symplectic matrix σ . Assuming that $V_t(P)$ is Frechet differentiable in P , its *symplectic invariance* (35) can be described in differential terms. To formulate the lemma below, the matrix $P \in \mathbb{S}_{2n}^+$ is split into blocks as

$$P := \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{matrix} \uparrow^n \\ \downarrow^n \end{matrix} = \begin{bmatrix} P_{\bullet 1} & P_{\bullet 2} \end{bmatrix} \begin{matrix} \uparrow^{2n} \\ \downarrow^{2n} \end{matrix} = \begin{bmatrix} P_{1\bullet} \\ P_{2\bullet} \end{bmatrix} \begin{matrix} \uparrow^n \\ \downarrow^n \end{matrix}, \quad (36)$$

where P_{11} is associated with the state variables of the plant, whilst P_{22} pertains to those of the controller. The \mathbb{S}_{2n} -valued Frechet derivative of the minimum cost function has an analogous partitioning

$$Q_t(P) := \partial_P V_t(P) = \begin{bmatrix} \partial_{P_{11}} V_t & \partial_{P_{12}} V_t/2 \\ \partial_{P_{21}} V_t/2 & \partial_{P_{22}} V_t \end{bmatrix}, \quad (37)$$

where the 1/2-factor takes into account the symmetry of P . Note that $Q_T \equiv 0$ since $V_T \equiv 0$ in view of (34). Associated with V_t is a map $H_t : \mathbb{S}_{2n}^+ \rightarrow \mathbb{R}^{2n \times 2n}$ defined by

$$H_t(P) := Q_t(P)P, \quad (38)$$

which is also partitioned into blocks as in (36) except that the matrix $H_t(P)$ is not necessarily symmetric. Also,

$$\mathbf{H}(N) := -J_0 \mathbf{S}(J_0 N) = -\mathbf{S}(N J_0) J_0 = (N + J_0 N^T J_0)/2 \quad (39)$$

denotes the orthogonal projection of a matrix $N \in \mathbb{R}^{n \times n}$ onto the subspace of Hamiltonian matrices, with \mathbf{S} the *symmetrizer* defined by

$$\mathbf{S}(N) := (N + N^T)/2. \quad (40)$$

Lemma 2 Suppose the minimum cost function $V_t(P)$ in (34) is Frechet differentiable with respect to $P \in \mathbb{S}_{2n}^+$. Then it satisfies the PDE

$$\mathbf{H}(H_t^{22}(P)) = 0, \quad (41)$$

which means that the controller block of the matrix $H_t(P)$ from (38) is a skew-Hamiltonian matrix.

Proof The transformations $P \mapsto SPS^T$ in (35) form a Lie group whose tangent space can be identified with the subspace of Hamiltonian matrices τ of order n . As the matrix exponential of a Hamiltonian matrix, $\sigma_\varepsilon := e^{\varepsilon\tau}$ is a symplectic matrix for any real ε . Therefore, if V_t is smooth, then by differentiating the left-hand side of (35) with $S_\varepsilon := \begin{bmatrix} I & 0 \\ 0 & \sigma_\varepsilon \end{bmatrix}$ as a composite function at $\varepsilon = 0$, it follows that the resulting Lie derivative [5] of V_t vanishes:

$$\begin{aligned} 0 &= \partial_\varepsilon V_t(S_\varepsilon P S_\varepsilon^T) \Big|_{\varepsilon=0} = 2 \left\langle \partial_P V_t, \begin{bmatrix} 0 \\ I \end{bmatrix} \tau P_{2\bullet} \right\rangle \\ &= 2 \left\langle \begin{bmatrix} 0 & I \end{bmatrix} Q_t P_{\bullet 2}, \tau \right\rangle = 2 \langle \mathbf{H}(H_t^{22}(P)), \tau \rangle. \end{aligned} \quad (42)$$

Here, the notations (36)–(38) are used. Since the relation (42) is valid for any Hamiltonian matrix τ , then (41) follows. ■

The relation (41) is, in fact, a system of first order scalar homogeneous linear PDEs which are associated with $n(n+1)/2$ entries of a real symmetric matrix of order n . This system is underdetermined since, for a fixed P_{11} , the total number of independent scalar variables is $n(3n+1)/2$. As we will show in Appendix A, this system of PDEs satisfies the involutivity condition and is locally completely integrable by the Frobenius integration theorem [5]. Instead of “disassembling” (41) into scalar equations which would lead to the loss of the underlying algebraic structure, it can be treated as one PDE with noncommutative matrix-valued variables. Such PDEs were encountered, for example, in entropy variational problems for Gaussian diffusion processes [20]. The general solution of the PDE (41) is given below.

Theorem 1 *Suppose $f_t : \mathbb{S}_n^+ \times \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$ is a continuously Frechet differentiable function. Then the function*

$$V_t(P) := f_t(P_{11}, P_{12}(P_{22}^{-1} + J_0)P_{21}), \quad (43)$$

defined for $P \in \mathbb{S}_{2n}^+$ with $P_{22} \succ 0$, satisfies the PDE (41). Moreover, (43) describes a general smooth solution of the PDE over any connected component of the set $\{P \in \mathbb{S}_{2n}^+ : \det P_{12} \neq 0, P_{22} \succ 0\}$.

The proof of Theorem 1 is given in Appendix A and employs ideas from the method of characteristics for conventional PDEs [3, 23] in combination with a nonlinear change of variables. The theorem shows that, due to the symplectic invariance, which holds at any time t , the minimum cost function $V_t(P)$ depends on the matrix P only through the special combinations of its blocks P_{11} , $P_{12}P_{22}^{-1}P_{21}$, $P_{12}J_0P_{21}$ which constitute a maximal set of nonconstant invariants of P with respect to the transformation group $P \mapsto SPS^T$ described in (35).

7 Hamilton-Jacobi-Bellman equation

Assuming that the minimum cost function $V_t(P)$ from (34) is continuously differentiable with respect to t and P in the sense of Frechet, the dynamic programming principle yields the HJBE

$$\partial_t V_t(P) + \inf_{u \in \mathbb{U}} \Pi_t(P, u, Q_t(P)) = 0. \quad (44)$$

Here, the minimization is over the triple $u := (b, e, R)$ of the current matrices $b := b_t$, $e := e_t$, $R := R_t$ of the PR controller, and the map $Q_t : \mathbb{S}_{2n}^+ \rightarrow \mathbb{S}_{2n}$ is associated with V_t by (37). Also,

$$\Pi_t(P, u, Q) := \langle \mathcal{C}_t^T \mathcal{C}_t, P \rangle + \langle Q, \mathcal{L}_{t,u}(P) \rangle \quad (45)$$

is the *control Hamiltonian* [19] of Pontryagin's minimum principle [15] applied to the CQLQG problem (30) as an optimal control problem for the dynamical system (32) with state P and control u . In view of (14), (27), (28) and (32), the matrix $R \in \mathbb{S}_n$, which parameterizes the free Hamiltonian operator of the PR controller, enters the control Hamiltonian $\Pi_t(P, b, e, R, Q)$ only through \mathcal{A}_t and in an affine fashion. Moreover, by considering (45) with $Q = Q_t(P)$ as in (44), it follows that

$$\begin{aligned} \Pi_t(P, b, e, R, Q_t(P)) &= \Pi_t(P, b, e, 0, Q_t(P)) + 2 \left\langle Q_t(P), \begin{bmatrix} 0 \\ I \end{bmatrix} J_0 R \begin{bmatrix} 0 & I \end{bmatrix} P \right\rangle \\ &= \Pi_t(P, b, e, 0, Q_t(P)) + 2 \left\langle \mathbf{H}(H_t^{22}(P)), J_0 R \right\rangle \\ &= \Pi_t(P, b, e, 0, Q_t(P)), \end{aligned} \quad (46)$$

where we have used the notations (36)–(38) and Lemma 2. The R -independence of the right-hand side of (46) reduces the minimization problem in (44) to

$$\inf_{u \in \mathbb{U}} \Pi_t(P, u, Q_t(P)) = \inf_{b, e} \Pi_t(P, b, e, 0, Q_t(P)). \quad (47)$$

This does not mean, however, that $R_t = 0$ has to be satisfied for the optimal quantum controller. The optimization problem (47) is solved in Section 8. We will now show that the map Q_t from (37), evaluated at an optimal trajectory of the system (32) and thus describing the costate of this system through the Pontryagin equations

$$\dot{P}_t = \partial_Q \Pi_t, \quad \dot{Q}_t = -\partial_P \Pi_t, \quad (48)$$

coincides with the observability Gramian of the closed-loop system (14). Here, $(\dot{})$ is the total time derivative, and the partial Frechet derivatives of (45) are taken with respect to Q , P as independent \mathbb{S}_{2n} -valued variables.

Lemma 3 *Suppose the minimum cost function $V_t(P)$ in (34) is twice continuously Frechet differentiable in t and P . Also, suppose there exist functions $b_t^\diamond(P)$ and $e_t^\diamond(P)$ which are Frechet differentiable in P and deliver the minimum in (47). Then the matrix $Q_t^\diamond := Q_t(P_t^\diamond)$, obtained by evaluating the map (37) at an optimal trajectory P_t^\diamond of the system (32), satisfies the differential Lyapunov equation*

$$\dot{Q}_t^\diamond = -\mathcal{A}_t^{\diamond T} Q_t^\diamond - Q_t^\diamond \mathcal{A}_t^\diamond - \mathcal{C}_t^{\diamond T} \mathcal{C}_t^\diamond, \quad (49)$$

where $\mathcal{A}_t^\diamond, \mathcal{C}_t^\diamond$ are the corresponding state-space matrices of the closed-loop system with the optimal CQLQG controller.

Proof The twice continuous differentiability of $V_t(P)$ ensures the interchangeability of its partial derivatives in t and P , so that $\partial_P \partial_t V_t = \partial_t \partial_P V_t = \partial_t Q_t$ in view of (37). Hence, by substituting (47) into (44) and differentiating the HJBE with respect to P , it follows that

$$\partial_t Q_t + d_P \Pi_t(P, b_t^\diamond, e_t^\diamond, 0, Q_t)/dP = 0. \quad (50)$$

Since the pair $(b_t^\diamond, e_t^\diamond)$ minimizes $\Pi_t(P, b, e, Q_t)$ in $(b, e) \in \mathbb{R}^{n \times m_2} \times \mathbb{R}^{n \times p_1}$ (that is, over an open set), then it is a critical point of this function, where both $\partial_b \Pi_t$ and $\partial_e \Pi_t$ vanish. Therefore,

$$\begin{aligned} d\Pi_t(P, b_t^\diamond, e_t^\diamond, 0, Q_t)/dP &= \partial_P \Pi_t + (\partial_P b_t^\diamond)^\dagger (\partial_b \Pi_t) + (\partial_P e_t^\diamond)^\dagger (\partial_e \Pi_t) + (\partial_P Q_t)^\dagger (\partial_Q \Pi_t) \\ &= \partial_P \Pi_t + (\partial_P Q_t)^\dagger (\partial_Q \Pi_t), \end{aligned} \quad (51)$$

where $\partial_P Q_t = \partial_P^2 V_t$ is a well-defined self-adjoint operator on \mathbb{S}_{2n} in view of (37) and the twice continuous Frechet differentiability of V_t . Now, from (32) and (45), it follows that

$$\partial_P \Pi_t = \mathcal{C}_t^T \mathcal{C}_t + (\partial_P \mathcal{L}_{t,u}(P))^\dagger(Q) = \mathcal{A}_t^T Q + Q \mathcal{A}_t + \mathcal{C}_t^T \mathcal{C}_t. \quad (52)$$

Since (45) implies that $\partial_Q \Pi_t = \mathcal{L}_{t,u}(P)$, then substitution of (51), (52) into (50) yields the PDE

$$\partial_t Q_t + \mathcal{A}_t^{\diamond T} Q_t + Q_t \mathcal{A}_t^\diamond + \mathcal{C}_t^{\diamond T} \mathcal{C}_t^\diamond + (\partial_P Q_t)^\dagger(\mathcal{L}_{t,u_t}(P)) = 0. \quad (53)$$

For the matrix P_t^\diamond governed by (32) with $u_t = (b_t^\diamond, e_t^\diamond, R_t)$, the matrix $(\partial_t Q_t)(P_t^\diamond) + (\partial_P Q_t)^\dagger(\mathcal{L}_{t,u_t}(P_t^\diamond)) = dQ_t(P_t^\diamond)/dt$ is the total time derivative of Q_t^\diamond . Hence, (53) leads to (49). ■

The ODE (49), whose right-hand side coincides with $-\partial_P \Pi_t$ in view of (52) and in conformance with (48), is the differential Lyapunov equation which governs the observability Gramian Q_t^\diamond of the closed-loop system under the optimal CQLQG controller. It is solved backwards in time $t \leq T$ with zero terminal condition $Q_T^\diamond = 0$. This (or an alternative reasoning involving the monotonicity of $V_t(P)$ in P), can be used to show that the map Q_t given by (37), takes values in \mathbb{S}_{2n}^+ . Therefore, $H_t(P)$ in (38) is a diagonalizable matrix with all real nonnegative eigenvalues which correspond to the squared Hankel singular values of the closed-loop system in view of the interpretation of Q_t and P_t as observability and controllability Gramians. We will refer to H_t as the *Hankelian* of the closed-loop system.

8 Optimal controller gain matrices

Since the matrix \mathcal{C}_t in (14) depends only on b_t in view of (28), the minimization on the right-hand side of (47) can be represented as

$$\inf_u \Pi_t(P, u, Q_t) = \inf_b \langle \mathcal{C}_t^T \mathcal{C}_t, P \rangle + \inf_e \langle Q_t, \mathcal{L}_{t,b,e,0}(P) \rangle \quad (54)$$

which is a repeated minimization problem over the PR controller gain matrices $b := b_t$ and $e := e_t$. Here,

$$\mathcal{L}_{t,b,e,0}(P) = \tilde{\mathcal{A}}_t P + P \tilde{\mathcal{A}}_t^T + \mathcal{B}_t \mathcal{B}_t^T \quad (55)$$

is the Lyapunov operator from (32) obtained by letting $R_t = 0$ in (27) and substituting the remaining skew-Hamiltonian part \tilde{a}_t of the controller matrix a_t into (14), which yields

$$\tilde{\mathcal{A}}_t := \begin{bmatrix} A_t & E_t c_t \\ e_t C_t & \tilde{a}_t \end{bmatrix}. \quad (56)$$

In turn, the repeated minimization problem (54) can be decoupled into two independent problems as follows. In view of the structure of the matrices \tilde{a}_t and c_t in (27) and (28), the matrix $\tilde{\mathcal{A}}_t$ from (56) is a quadratic function of the controller gain matrices b, e . The dependencies of $\tilde{\mathcal{A}}_t$ on b and e can be isolated as

$$\tilde{\mathcal{A}}_t = \underbrace{\begin{bmatrix} A_t & 0 \\ 0 & 0 \end{bmatrix}}_{\mathcal{A}_t^0} + \underbrace{\begin{bmatrix} 0 & E_t d_t J_2 b^T J_0 \\ 0 & b J_2 b^T J_0 / 2 \end{bmatrix}}_{\tilde{\mathcal{A}}_t} + \underbrace{\begin{bmatrix} 0 & 0 \\ e C_t & e D_t J_1 D_t^T e^T J_0 / 2 \end{bmatrix}}_{\tilde{\mathcal{A}}_t}, \quad (57)$$

where the matrix \mathcal{A}_t^0 is independent of both b and e , whilst $\check{\mathcal{A}}_t$ only depends on b and $\hat{\mathcal{A}}_t$ only depends on e . In a similar vein, (14) and (28) imply that

$$\mathcal{B}_t \mathcal{B}_t^T = \underbrace{\begin{bmatrix} B_t B_t^T + E_t d_t d_t^T E_t^T & 0 \\ 0 & 0 \end{bmatrix}}_{\Gamma_t^0} + \underbrace{\begin{bmatrix} 0 & E_t d_t b^T \\ b d_t^T E_t^T & b b^T \end{bmatrix}}_{\check{\Gamma}_t} + \underbrace{\begin{bmatrix} 0 & B_t D_t^T e^T \\ e D_t B_t^T & e D_t D_t^T e^T \end{bmatrix}}_{\hat{\Gamma}_t}, \quad (58)$$

$$C_t^T C_t = \underbrace{\begin{bmatrix} F_t^T F_t & 0 \\ 0 & 0 \end{bmatrix}}_{\Delta_t^0} + \underbrace{\begin{bmatrix} 0 & F_t^T G_t d_t J_2 b^T J_0 \\ J_0 b J_2 d_t^T G_t^T F_t & J_0 b J_2 d_t^T G_t^T G_t d_t J_2 b^T J_0 \end{bmatrix}}_{\check{\Delta}_t}, \quad (59)$$

where Γ_t^0, Δ_t^0 are independent of both b and e , the matrices $\check{\Gamma}_t, \check{\Delta}_t$ only depend on b , whilst $\hat{\Gamma}_t$ only depends on e . By substituting (57), (58) into (55) and combining the result with (59), the repeated minimization problem in (54) is indeed split into

$$\begin{aligned} \inf_u \Pi_t(P, u, Q_t) &= \langle \Delta_t^0, P \rangle + \langle Q_t, 2\mathcal{A}_t^0 P + \Gamma_t^0 \rangle \\ &\quad + \inf_b (\langle \check{\Delta}_t, P \rangle + \langle Q_t, 2\check{\mathcal{A}}_t P + \check{\Gamma}_t \rangle) \\ &\quad + \inf_e \langle Q_t, 2\hat{\mathcal{A}}_t P + \hat{\Gamma}_t \rangle. \end{aligned} \quad (60)$$

Both minimization problems on the right-hand side of (60) are quadratic optimization problems whose solutions are available in closed form and lead to the optimal values for the controller gain matrices b and e . The fact that these problems are independent describes a *quasi-separation* property of the gain matrices [22] and can be interpreted as a weaker quantum counterpart of the filtering/control separation principle of the classical LQG control. We will first consider the minimization with respect to the controller observation gain matrix e . From (57) and (58), it follows that

$$\begin{aligned} \langle Q_t, 2\hat{\mathcal{A}}_t P + \hat{\Gamma}_t \rangle &= \left\langle Q_t, \begin{bmatrix} 0 & 0 \\ 2e C_t & e D_t J_1 D_t^T e^T J_0 \end{bmatrix} P + \begin{bmatrix} 0 & B_t D_t^T e^T \\ e D_t B_t^T & e D_t D_t^T e^T \end{bmatrix} \right\rangle \\ &= \langle 2(H_t^{21} C_t^T + Q_t^{21} B_t D_t^T) + \mathfrak{M}_t(e), e \rangle, \end{aligned} \quad (61)$$

where

$$\mathfrak{M}_t := \llbracket H_t^{22} J_0, D_t J_1 D_t^T \mid Q_t^{22}, D_t D_t^T \rrbracket \quad (62)$$

is a self-adjoint operator of grade two (see Appendix B) on $\mathbb{R}^{n \times p_1}$. Here, we have used the property that the matrix $H_t^{22} J_0$ is antisymmetric since the controller block H_t^{22} of the Hankelian (38) is skew-Hamiltonian in view of (41). If the operator \mathfrak{M}_t is positive definite, then the quadratic function on the right-hand side of (61) achieves its minimum value

$$\min_e \langle Q_t, 2\hat{\mathcal{A}}_t P + \hat{\Gamma}_t \rangle = -\|H_t^{21} C_t^T + Q_t^{21} B_t D_t^T\|_{\mathfrak{M}_t^{-1}}^2 \quad (63)$$

at a unique point

$$e_t^\diamond := -\mathfrak{M}_t^{-1}(H_t^{21} C_t^T + Q_t^{21} B_t D_t^T). \quad (64)$$

Here, for a positive definite self-adjoint operator \mathfrak{D} on the Hilbert space $\mathbb{R}^{p \times q}$ with the standard Frobenius inner product $\langle \cdot, \cdot \rangle$, we denote by

$$\|N\|_{\mathfrak{D}} := \sqrt{\langle N, N \rangle_{\mathfrak{D}}}$$

the norm of a matrix $N \in \mathbb{R}^{p \times q}$ associated with the “weighted” Frobenius inner product

$$\langle K, N \rangle_{\mathcal{D}} := \langle K, \mathcal{D}(N) \rangle.$$

The minimization in (60) with respect to the controller noise gain matrix b is performed in a similar fashion. It follows from (57)–(59) that

$$\begin{aligned} & \langle \check{\Delta}_t, P \rangle + \langle Q_t, 2\check{\mathcal{A}}_t P + \check{\Gamma}_t \rangle \\ &= 2\langle P_{21}, J_0 b J_2 d_t^T G_t^T F_t \rangle + \langle P_{22}, J_0 b J_2 d_t^T G_t^T G_t d_t J_2 b^T J_0 \rangle \\ &+ \left\langle Q_t, \begin{bmatrix} 0 & 2E_t d_t J_2 b^T J_0 \\ 0 & b J_2 b^T J_0 \end{bmatrix} P + \begin{bmatrix} 0 & E_t d_t b^T \\ b d_t^T E_t^T & b b^T \end{bmatrix} \right\rangle \\ &= \left\langle 2(Q_t^{21} E_t d_t + J_0 ((H_t^{12})^T E_t + P_{21} F_t^T G_t) d_t J_2) + \mathfrak{N}_t(b), b \right\rangle, \end{aligned} \quad (65)$$

where

$$\mathfrak{N}_t := \llbracket H_t^{22} J_0, J_2 \mid Q_t^{22}, I \mid J_0 P_{22} J_0, J_2 d_t^T G_t^T G_t d_t J_2 \rrbracket \quad (66)$$

is a self-adjoint operator of grade three (see Appendix B) on $\mathbb{R}^{n \times m_2}$ in view of the antisymmetry of $H_t^{22} J_0$. If \mathfrak{N}_t is positive definite, then the quadratic function of b , given by (65), achieves its minimum value

$$\begin{aligned} & \min_b (\langle \check{\Delta}_t, P \rangle + \langle Q_t, 2\check{\mathcal{A}}_t P + \check{\Gamma}_t \rangle) \\ &= -\|Q_t^{21} E_t d_t + J_0 ((H_t^{12})^T E_t + P_{21} F_t^T G_t) d_t J_2\|_{\mathfrak{N}_t^{-1}}^2 \end{aligned} \quad (67)$$

at a unique point

$$b_t^\diamond := -\mathfrak{N}_t^{-1} (Q_t^{21} E_t d_t + J_0 ((H_t^{12})^T E_t + P_{21} F_t^T G_t) d_t J_2). \quad (68)$$

Finally, by substituting (63), (67) into (60) and using the representation

$$\langle \Delta_t^0, P \rangle + \langle Q_t, 2\mathcal{A}_t^0 P + \Gamma_t^0 \rangle = \langle F_t^T F_t, P_{11} \rangle + 2\langle H_t^{11}, A_t \rangle + \langle Q_t^{11}, B_t B_t^T + E_t d_t d_t^T E_t^T \rangle,$$

which follows from (57)–(59), the HJBE (44) is reduced to the Hamilton-Jacobi equation (HJE) below.

Theorem 2 *Suppose the minimum cost function $V_t(P)$ for the CQLQG problem, defined by (34), is continuously Frechet differentiable in t and P , and the associated self-adjoint operators \mathfrak{M}_t and \mathfrak{N}_t in (62) and (66) are positive definite. Then the function $V_t(P)$ satisfies the HJE*

$$\begin{aligned} & \partial_t V_t + \langle F_t^T F_t, P_{11} \rangle + 2\langle H_t^{11}, A_t \rangle + \langle Q_t^{11}, B_t B_t^T + E_t d_t d_t^T E_t^T \rangle \\ & - \|Q_t^{21} E_t d_t + J_0 ((H_t^{12})^T E_t + P_{21} F_t^T G_t) d_t J_2\|_{\mathfrak{N}_t^{-1}}^2 \\ & - \|H_t^{21} C_t^T + Q_t^{21} B_t D_t^T\|_{\mathfrak{M}_t^{-1}}^2 = 0, \end{aligned}$$

and the gain matrices $b_t^\diamond, e_t^\diamond$ of an optimal PR controller are computed according to (68) and (64).

By using [22, Lemma 5], it can be shown, that if the controller block Q_t^{22} of the closed-loop system observability Gramian is nonsingular, the matrix D_t is of full row rank and $\mathbf{r}((Q_t^{22})^{-1}H_t^{22}J_0) < 1$, with $\mathbf{r}(\cdot)$ the spectral radius of a matrix, then both operators \mathfrak{M}_t and \mathfrak{N}_t are positive definite. Since each of the matrices $Q_t(P)$ and P enters \mathfrak{M}_t and \mathfrak{N}_t in (62) and (66) in a linear fashion, the dependence of b_t^\diamond and e_t^\diamond on these matrices is linear-fractional and hence, smooth, provided $\mathfrak{M}_t \succ 0$ and $\mathfrak{N}_t \succ 0$ (such values of P form an open set). Therefore, if, in addition to the assumptions of Theorem 2, the minimum cost function $V_t(P)$ is *twice* continuously Frechet differentiable with respect to P , then the optimal CQLQG controller gain matrices b_t^\diamond and e_t^\diamond are continuously Frechet differentiable functions of P . If, furthermore, V_t is twice continuously Frechet differentiable in t and P , this ensures the applicability of Lemma 3, which utilizes the viewpoint of Pontryagin's minimum principle on the CQLQG problem.

9 Equations for the optimal quantum controller

The set of equations for the optimal CQLQG controller over the time interval $0 \leq t \leq T$ consists of two Lyapunov ODEs (32) and (49) for the controllability and observability Gramians P_t, Q_t of the closed-loop system:

$$\dot{P}_t = \mathcal{A}_t P_t + P_t \mathcal{A}_t^T + \mathcal{B}_t \mathcal{B}_t^T, \quad (69)$$

$$\dot{Q}_t = -\mathcal{A}_t^T Q_t - Q_t \mathcal{A}_t - \mathcal{C}_t^T \mathcal{C}_t, \quad (70)$$

with the split boundary conditions $P_0 = P$ and $Q_T = 0$, where $P \in \mathbb{S}_{2n}^+$ is a given matrix satisfying $P + i\Theta_0/2 \succ 0$. According to (14), (27), (28), the closed-loop system matrices $\mathcal{A}_t, \mathcal{B}_t, \mathcal{C}_t$ are expressed in terms of the controller matrices b_t, e_t, R_t as

$$\mathcal{A}_t := \begin{bmatrix} A_t & E_t d_t J_2 b_t^T J_0 \\ e_t C_t & (e_t D_t J_1 D_t^T e_t^T + b_t J_2 b_t^T) J_0/2 + J_0 R_t \end{bmatrix}, \quad (71)$$

$$\mathcal{B}_t := \begin{bmatrix} B_t & E_t d_t \\ e_t D_t & b_t \end{bmatrix}, \quad (72)$$

$$\mathcal{C}_t := [F_t \quad G_t d_t J_2 b_t^T J_0]. \quad (73)$$

In turn, the optimal controller gain matrices b_t, e_t are completely specified by the Gramians P_t, Q_t (which determine the closed-loop system Hankelian H_t) according to (64), (68) as

$$e_t := -\llbracket H_t^{22} J_0, D_t J_1 D_t^T \mid Q_t^{22}, D_t D_t^T \rrbracket^{-1} (H_t^{21} C_t^T + Q_t^{21} B_t D_t^T), \quad (74)$$

$$b_t := -\llbracket H_t^{22} J_0, J_2 \mid Q_t^{22}, I \mid J_0 P_{22} J_0, J_2 d_t^T G_t^T G_t d_t J_2 \rrbracket^{-1} (Q_t^{21} E_t d_t + J_0 ((H_t^{12})^T E_t + P_{21} F_t^T G_t) d_t J_2), \quad (75)$$

where the inverses of the special self-adjoint operators can be represented through the vectorization of matrices; see Appendix B. Therefore, the set of equations for the optimal CQLQG controller is a split boundary value problem for two Lyapunov ODEs (69), (70) which are nonlinearly coupled through the algebraic equations (71)–(75). The matrix R_t , which affinely enters the right-hand side of these ODEs through the matrix \mathcal{A}_t in (71), appears to be a free parameter in the sense that an equation for its optimal value is missing and

the optimal controller gain matrices e_t and b_t in (74) and (75) do not depend on the current value of R_t . Moreover, by using the identity

$$\begin{aligned}\dot{H}_t &= \dot{Q}_t P_t + Q_t \dot{P}_t \\ &= -(\mathcal{A}_t^T Q_t + Q_t \mathcal{A}_t + \mathcal{C}_t^T \mathcal{C}_t) P_t + Q_t (\mathcal{A}_t P_t + P_t \mathcal{A}_t^T + \mathcal{B}_t \mathcal{B}_t^T) \\ &= [H_t, \mathcal{A}_t^T] + Q_t \mathcal{B}_t \mathcal{B}_t^T - \mathcal{C}_t^T \mathcal{C}_t P_t\end{aligned}$$

(see also [21, Appendix C]), it can be shown that the skew-Hamiltonian structure of H_t^{22} in (41), trivially ensured at $T = 0$ by the terminal condition $Q_T = 0$, is preserved by the dynamics (69)–(75) for $t < T$ regardless of the choice of R_t . However, the function $[0, T] \ni t \mapsto R_t \in \mathbb{S}_n$ is responsible for the fulfillment of the split boundary conditions.

10 Concluding remarks

We have considered a time-varying Coherent Quantum LQG control problem which seeks a physically realizable quantum controller to minimize the finite-horizon LQG cost, and outlined a novel approach towards its solution. Using the Hamiltonian parameterization of PR controllers, which relates them to open quantum harmonic oscillators, we have recast the CQLQG problem as a covariance control problem. Dynamic programming and Pontryagin’s minimum principle have been applied to the resulting optimal control problem for a subsidiary deterministic dynamical system whose state is the symmetric part of the quantum covariance matrix of the closed-loop system state vector governed by a differential Lyapunov equation. It has been shown that the corresponding costate is the observability Gramian of the closed-loop system. By using the invariance of the minimum cost function under the group of symplectic similarity transformations of PR controllers, we have derived algebraic equations for the gain matrices of the optimal CQLQG controller and established their partial decoupling as a weaker quantum analogue of the classical LQG control/filtering separation principle. These equations express the optimal controller gain matrices in terms of the current observability and controllability Gramians of the closed-loop system thus leading to a split boundary value problem for two nonlinearly coupled differential Lyapunov equations. The difficulty of solving this problem lies in the coupling of the differential equations and mixed nature of the boundary conditions. However, the special structure of the minimum cost function, enforced by the symplectic invariance, suggests the possibility of reducing the order of these equations by nonlinear transformation of the blocks of the Gramians. Another resource yet to be explored is to consider the CQLQG problem for PR plants. The existence/uniqueness of solutions to the equations for the state-space realization matrices of the optimal CQLQG controller remains an open problem and so do their possible reduction and numerical implementation. These issues are a subject of current research and will be reported in subsequent publications.

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A Proof of Theorem 1

By omitting the dependence of the minimum cost function $V_t(P)$ on t and P_{11} which are assumed to be fixed, and introducing the variables

$$X := P_{12} = P_{21}^T, \quad Y := P_{22}, \quad (76)$$

the PDE (41) takes the form

$$\mathbf{H}(M(V)) = 0. \quad (77)$$

Here, use is made of (36)–(40), and M is a linear differential operator which maps a Frechet differentiable function $\mathbb{R}^{n \times n} \times \mathbb{S}_n \ni (X, Y) \mapsto v(X, Y) \in \mathbb{R}$ to an $\mathbb{R}^{n \times n}$ -valued function $M(v)$ defined on the same domain by

$$M(v) := \frac{1}{2} X^T \partial_X v + Y \partial_Y v. \quad (78)$$

The next section verifies the involutivity of the PDE (77) as a system of scalar PDEs. Then we consider two particular solutions of this PDE in Section A.2 which allow its general solution to be obtained in Section A.3 through a change-of-variables technique.

A.1 Verification of involutivity

We will now verify the fulfillment of the local complete integrability conditions for the PDE (77), which in view of (39) and (40) is equivalent to

$$\mathbf{S}(M(V)J_0) = 0 \quad (79)$$

whose left-hand side is a real symmetric matrix of order n . If the complete integrability holds, then the PDE has a n^2 -dimensional integral manifold which can be represented using an $\mathbb{R}^{n \times n}$ -valued function of the matrices X and Y . For any constant matrix $Z \in \mathbb{S}_n$, let L_Z be a linear operator which maps a Frechet differentiable function $\mathbb{R}^{n \times n} \times \mathbb{S}_n \ni (X, Y) \mapsto v(X, Y) \in \mathbb{R}$ to a function $L_Z(v) : \mathbb{R}^{n \times n} \times \mathbb{S}_n \rightarrow \mathbb{R}$ defined by

$$L_Z(v) = \langle Z, \mathbf{S}(M(v)J_0) \rangle = -\langle ZJ_0, M(v) \rangle = -A_{ZJ_0}(v). \quad (80)$$

Here, M is given by (78), and A_N is a linear differential operator, which is associated with a matrix $N \in \mathbb{R}^{n \times n}$ and maps the function v to another function of X, Y as

$$A_N(v) := \langle N, M(v) \rangle. \quad (81)$$

Thus, the operators L_Z , associated with symmetric matrices Z by (80), are the operators A_N considered for Hamiltonian matrices N , although, in general, the matrix N in (81) can be arbitrary. By the Frobenius integration theorem [5] (see also, [1, pp. 158–165]), the local complete integrability of the PDE (79) will be proved if we show that for any constant matrices $Z_1, Z_2 \in \mathbb{S}_n$, there exists a matrix $Z_3 \in \mathbb{S}_n$, which is allowed to depend on X and Y and such that

$$[L_{Z_1}, L_{Z_2}](v) = L_{Z_3}(v) \quad (82)$$

is satisfied for any twice continuously Frechet differentiable function v described above. The relation (82) is an inner product form of the involutivity condition for the PDE (79) regarded as a system of scalar PDEs, with the inner product used to represent linear combinations of the individual equations in a coordinate-free fashion.

Lemma 4 *For any constant matrices $N_1, N_2 \in \mathbb{R}^{n \times n}$, the operators (81), considered on twice continuously Frechet differentiable test functions v , satisfy the commutation relation*

$$[A_{N_1}, A_{N_2}] = A_{[N_1, N_2]/2}. \quad (83)$$

Proof For any $N \in \mathbb{R}^{n \times n}$ and any twice continuously Frechet differentiable function $v : \mathbb{R}^{n \times n} \times \mathbb{S}_n \mapsto \mathbb{R}$, the function $A_N(v)$ is continuously Frechet differentiable and its derivatives are computed as

$$\begin{aligned} \partial_X A_N(v) &= \frac{1}{2}(\partial_X v N^T + \partial_X^2 v(XN)) + \partial_Y \partial_X v(YN), \\ \partial_Y A_N(v) &= \frac{1}{2}(N \partial_Y v + \partial_Y v N^T + \partial_Y^2 v(YN + N^T Y) + \partial_X \partial_Y v(XN)), \end{aligned}$$

where the relation $\partial_X \partial_Y v = (\partial_Y \partial_X v)^\dagger$ and self-adjointness of the linear operators $\partial_X^2 v$ and $\partial_Y^2 v$ are used. Hence, the composition of the differential operators (81), associated with $N_1, N_2 \in \mathbb{R}^{n \times n}$, is computed as

$$\begin{aligned} A_{N_1}(A_{N_2}(v)) &= \frac{1}{2}\langle XN_1, \partial_X A_{N_2}(v) \rangle + \langle YN_1, \partial_Y A_{N_2}(v) \rangle \\ &= \frac{1}{4}\langle XN_1N_2, \partial_X v \rangle + \frac{1}{4}\langle N_2^T YN_1 + N_1^T YN_2 + YN_1N_2 + N_2^T N_1^T Y, \partial_Y v \rangle \\ &\quad + \frac{1}{2}\langle XN_1, \partial_X^2 v(XN_2) \rangle + \frac{1}{2}\langle XN_1, \partial_Y \partial_X v(YN_2) \rangle \\ &\quad + \frac{1}{2}\langle YN_1, \partial_X \partial_Y v(XN_2) \rangle + \frac{1}{4}\langle YN_1 + N_1^T Y, \partial_Y^2 v(YN_2 + N_2^T Y) \rangle. \end{aligned} \quad (84)$$

Note that the part of the right-hand side of (84), which involves the second-order derivatives of v , is invariant under the transposition $(N_1, N_2) \mapsto (N_2, N_1)$, and so also is the matrix $N_2^T YN_1 + N_1^T YN_2$. Therefore, the commutator of A_{N_1} and A_{N_2} takes the form

$$\begin{aligned} [A_{N_1}, A_{N_2}](v) &= \frac{1}{4}\langle X[N_1, N_2], \partial_X v \rangle + \frac{1}{2}\langle Y[N_1, N_2], \partial_Y v \rangle \\ &= \frac{1}{2}\langle [N_1, N_2], M(v) \rangle = A_{[N_1, N_2]/2}(v), \end{aligned}$$

which holds for twice continuously Frechet differentiable functions v , thus establishing (83). ■

In view of the Frobenius theorem mentioned above, Lemma 4 implies the local complete integrability of the PDE

$$M(V) = 0, \quad (85)$$

whose left-hand side is given by (78). The solutions of (85) are also solutions of the PDE (77), but not visa versa. Since (85) is a system of n^2 independent scalar PDEs, the integrability suggests that it has a $n(n+1)/2$ -dimensional integral manifold which can be represented using an \mathbb{S}_n -valued map. Moreover, Lemma 4 also establishes the involutivity for the PDE (77), with its n^2 -dimensional integral manifold representable by an $\mathbb{R}^{n \times n}$ -valued map, or a pair of maps with values in \mathbb{S}_n and \mathbb{A}_n , where $\mathbb{A}_n := \mathbb{S}_n^\perp$ is the subspace of real antisymmetric matrices of order n , which is the orthogonal complement of the subspace \mathbb{S}_n in the sense of the Frobenius inner product in $\mathbb{R}^{n \times n}$. Indeed,

$$[Z_1 J_0, Z_2 J_0] = (Z_1 J_0 Z_2 - Z_2 J_0 Z_1) J_0$$

for any $Z_1, Z_2 \in \mathbb{S}_n$, in accordance with the fact that the commutator of Hamiltonian matrices is also a Hamiltonian matrix. Therefore, by applying (83) to the operators (80), it follows that the involutivity condition (82) holds with $Z_3 := (Z_2 J_0 Z_1 - Z_1 J_0 Z_2)/2 \in \mathbb{S}_n$, which implies the local complete integrability for the PDE (77).

A.2 Two particular solutions

Lemma 5 *Suppose $f : \mathbb{S}_n \rightarrow \mathbb{R}$ is a Frechet differentiable function. Then the function*

$$V(X, Y) := f(XY^{-1}X^T) \quad (86)$$

defined for $X \in \mathbb{R}^{n \times n}$ and $Y \in \mathbb{S}_n$ with $\det Y \neq 0$, satisfies the PDE (85). Moreover, (86) describes the general smooth solution of (85) on every connected component of the set $\{(X, Y) \in \mathbb{R}^{n \times n} \times \mathbb{S}_n : \det(XY) \neq 0\}$.

Proof With the matrices $X \in \mathbb{R}^{n \times n}$ and $Y \in \mathbb{S}_n$, where $\det Y \neq 0$, we associate the matrices

$$U := XY^{-1}, \quad W := XY^{-1}X^T. \quad (87)$$

The Frechet derivatives of W with respect to X and Y are expressed in terms of special linear operators of grade one (see Appendix B) and the matrix transpose operator \mathcal{T} as

$$\partial_X W = \llbracket I, U^T \rrbracket + \llbracket U, I \rrbracket \mathcal{T}, \quad \partial_Y W = -\llbracket U, U^T \rrbracket, \quad (88)$$

where the composition $M \circ N$ of linear operators M and N is written briefly as MN . Indeed, the first variation of the matrix-valued map $(X, Y) \mapsto W$ in (87) is computed as

$$\begin{aligned} \delta W &= (\delta X)Y^{-1}X^T + XY^{-1}(\delta X)^T - XY^{-1}(\delta Y)Y^{-1}X^T \\ &= (\delta X)U^T + U\delta X^T - U(\delta Y)U^T, \end{aligned} \quad (89)$$

which implies (88). The Frechet derivatives of the composite function $V = f \circ W$ from (86) are

$$\partial_X V = (\partial_X W)^\dagger(f') = (\llbracket I, U \rrbracket + \mathcal{T}\llbracket U^T, I \rrbracket)(f') = 2f'U, \quad (90)$$

where f' is the \mathbb{S}_n -valued Frechet derivative of the function f , and the relations $\llbracket \alpha, \beta \rrbracket^\dagger = \llbracket \alpha^T, \beta^T \rrbracket$ and $\mathcal{T}^\dagger = \mathcal{T}$ are used, with $(\cdot)^\dagger$ the adjoint with respect to the Frobenius inner product of matrices. By a similar reasoning,

$$\partial_Y V = (\partial_Y W)^\dagger(f') = -U^T f'U. \quad (91)$$

Substitution of (90) and (91) into the left-hand side of (85) yields

$$M(V) = (X^T - YU^T)f'U = 0, \quad (92)$$

so that the function V given by (86) indeed satisfies the PDE. We will now show that (86) is, in fact, the general smooth solution of the PDE (85) under the additional condition $\det X \neq 0$, in which case both U and W in (87) are nonsingular. To this end, using the ideas of the method of characteristics for conventional PDEs

[3,23], we will prove that any smooth function V satisfying the PDE (85) is constant on every connected component of the preimage

$$W^{-1}(S) := \{(X, Y) \in \mathbb{R}^{n \times n} \times \mathbb{S}_n : \det Y \neq 0, XY^{-1}X^T = S\} \quad (93)$$

of any given nonsingular matrix $S \in \mathbb{S}_n$ under the map $(X, Y) \mapsto W$, with W^{-1} the functional inverse. Indeed, let $[0, 1] \ni s \mapsto (X, Y) \in W^{-1}(S)$ be a smooth curve lying in this set. By differentiating the map W along such a curve and using (89), it follows that $0 = \dot{W} = \dot{X}U^T + U\dot{X}^T - U\dot{Y}U^T$, which, in view of $\det U \neq 0$, allows \dot{Y} to be expressed in terms of \dot{X} as

$$\dot{Y} = U^{-1}\dot{X} + \dot{X}^T U^{-T}, \quad (94)$$

where $(\dot{}) := \partial_s$. Hence, differentiation of V as a composite function along the curve yields

$$\begin{aligned} \dot{V} &= \langle \partial_X V, \dot{X} \rangle + \langle \partial_Y V, \dot{Y} \rangle \\ &= \langle \partial_X V, \dot{X} \rangle + \langle \partial_Y V, U^{-1}\dot{X} + \dot{X}^T U^{-T} \rangle \\ &= \langle \partial_X V + 2U^{-T}\partial_Y V, \dot{X} \rangle, \end{aligned} \quad (95)$$

where use is made of (94) and the symmetry of the matrix $\partial_Y V$. Since the PDE (85) implies that $\partial_X V + 2U^{-T}\partial_Y V = 2X^{-T}M(V) = 0$, then (95) yields $\dot{V} = 0$. Hence, every smooth solution V of (85) is constant over any connected component of the set $W^{-1}(S)$ from (93). Indeed, existence of two distinct points, which are connected by a smooth curve in $W^{-1}(S)$ and such that V takes different values at these endpoints, would contradict the constancy of V along any such curve established above. Thus, $V(X, Y)$ can only depend on X and Y through their special combination $XY^{-1}X^T$, and the ODE (94) generates characteristic curves on which smooth solutions of the PDE are constant. ■

Note that since (86) involves the matrix inverse Y^{-1} , the explicit representation of the solution would be hard to guess by treating (85) as a system of scalar PDEs.

Lemma 6 Suppose $f : \mathbb{A}_n \rightarrow \mathbb{R}$ is a Frechet differentiable function. Then the function $V : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$ defined by

$$V(X) := f(XJ_0X^T), \quad (96)$$

with J_0 the canonical antisymmetric matrix of order n from (24), satisfies the PDE

$$\mathbf{S}(X^T \partial_X V J_0) = 0. \quad (97)$$

Moreover, (96) describes the general solution of (97) among Frechet differentiable functions of X over any connected component of the set $\det X \neq 0$.

Proof Since

$$\partial_X (XJ_0X^T) = \llbracket I, J_0X^T \rrbracket + \llbracket XJ_0, I \rrbracket \mathcal{T},$$

then differentiation of (96) as a composite function of X yields

$$\partial_X V = -f'XJ_0 - (J_0X^T f')^T = -2f'XJ_0,$$

where we have also used the antisymmetry of the matrix f' . Hence, $X^T \partial_X V J_0 = 2X^T f'X$ is antisymmetric whence (97) follows. Now, to prove the converse, let $[0, 1] \ni s \mapsto X \in \mathbb{R}^{n \times n}$ be an arbitrary smooth curve in the set $\{X \in \mathbb{R}^{n \times n} : XJ_0X^T = \Omega\}$, where $\Omega \in \mathbb{A}_n$ is a given nonsingular antisymmetric matrix. By differentiating XJ_0X^T along such a curve, it follows that

$$\dot{X}J_0X^T + XJ_0\dot{X}^T = 0. \quad (98)$$

Since $\det X \neq 0$, then the left multiplication of (98) by X^{-1} and right multiplication by X^{-T} yields

$$\mathbf{A}(X^{-1}\dot{X}J_0) = 0, \quad (99)$$

with \mathbf{A} the *antisymmetrizer* defined by the orthogonal projection onto the subspace \mathbb{A}_n of real antisymmetric matrices of order n as

$$\mathbf{A}(N) := N - \mathbf{S}(N) = (N - N^T)/2.$$

In view of (39), the relation (99) is equivalent to the matrix $X^{-1}\dot{X}$ being Hamiltonian. Now, if V is an arbitrary smooth solution of the PDE (97), then its derivative along the curve is

$$\begin{aligned}\dot{V} &= \langle \partial_X V, \dot{X} \rangle = \langle X^T \partial_X V J_0, X^{-1} \dot{X} J_0 \rangle \\ &= \langle \mathbf{S}(X^T \partial_X V J_0), \mathbf{S}(X^{-1} \dot{X} J_0) \rangle \\ &+ \langle \mathbf{A}(X^T \partial_X V J_0), \mathbf{A}(X^{-1} \dot{X} J_0) \rangle = 0,\end{aligned}\quad (100)$$

where the Frobenius inner product is partitioned according to the orthogonal decomposition $\mathbb{R}^{n \times n} = \mathbb{S}_n \oplus \mathbb{A}_n$. In view of (100), the solution V is constant over any connected component of the set where $X J_0 X^T$ is a given nonsingular matrix, thus implying the representation (96). ■

A.3 General solution

The following theorem shows that the particular solutions of the PDE (77), obtained in the previous section, can be “assembled” into the general solution.

Theorem 3 *Suppose $f : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$ is a Frechet differentiable function. Then the function*

$$V(X, Y) := f(X(Y^{-1} + J_0)X^T), \quad (101)$$

defined for $X \in \mathbb{R}^{n \times n}$ and $Y \in \mathbb{S}_n$, with $\det Y \neq 0$, satisfies the PDE (77). Moreover, (101) is a general smooth solution of the PDE over any connected component of the set $\{(X, Y) \in \mathbb{R}^{n \times n} \times \mathbb{S}_n : \det(XY) \neq 0\}$.

Proof Since the function (101) can be represented as $V = g(XY^{-1}X^T, XJ_0X^T)$, where $g : \mathbb{S}_n \times \mathbb{A}_n \rightarrow \mathbb{R}$ is another Frechet differentiable function given by

$$g(\sigma, \omega) := f(\sigma + \omega), \quad (102)$$

then the first claim of the theorem follows from the corresponding statements of Lemmas 5 and 6. The fulfillment of the PDE (77) for the function (101) can also be verified directly using its partial Frechet derivatives

$$\partial_X V = 2(\mathbf{S}(f')XY^{-1} - \mathbf{A}(f')XJ_0), \quad (103)$$

$$\partial_Y V = -Y^{-1}X^T\mathbf{S}(f')XY^{-1}, \quad (104)$$

which follow from the relations $\partial_\sigma g = \mathbf{S}(f')$ and $\partial_\omega g = \mathbf{A}(f')$ for the function g in (102). Now, to prove that (101) is, in fact, the general solution of the PDE over any connected component of the set $\det(XY) \neq 0$, we employ the transformation $(X, Y) \mapsto (X, W)$, with W given by (87). This is a diffeomorphism since, for any nonsingular X , the matrix Y is uniquely and smoothly recovered from W as $Y = X^T W^{-1} X$. The action of the operator M from (78) on the function $h(X, W) := V(X, Y)$ written in the new independent variables X and W takes the form

$$\begin{aligned}M(V) &= \frac{1}{2}X^T(\partial_X h + (\partial_X W)^\dagger(\partial_W h)) \\ &+ X^T W^{-1}X(\partial_Y W)^\dagger(\partial_W h) = \frac{1}{2}X^T \partial_X h,\end{aligned}\quad (105)$$

where the terms containing $\partial_W h$ cancel each other due to the structure of the operators $\partial_X W$ and $\partial_Y W$ from (88) employed in the proof of Lemma 5. Substitution of (105) into the PDE (77) leads to the PDE $\mathbf{S}(X^T \partial_X h J_0) = 0$. By considering this last PDE for a fixed but otherwise arbitrary nonsingular $W \in \mathbb{S}_n$, and applying Lemma 6, it follows that its general solution over any connected component of the set $\det X \neq 0$ is described by $h(X, W) = \varphi(W, XJ_0X^T)$, where $\varphi : \mathbb{S}_n \times \mathbb{A}_n \rightarrow \mathbb{R}$ is a Frechet differentiable function. Since any such φ can be identified with a Frechet differentiable function $f : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$ by $f(N) = \varphi(\mathbf{S}(N), \mathbf{A}(N))$, cf. (102), this proves the second claim of Theorem 3. ■

Finally, Theorem 1 is obtained by applying Theorem 3 to the minimum cost function $V_t(P)$ for fixed but otherwise arbitrary t and P_{11} , assuming its Frechet smoothness on the set where the blocks of the covariance matrix P from (36) satisfy $\det P_{12} \neq 0$ and $P_{22} \succ 0$.

B Special linear operators on matrices

Following [22], we define, for any matrices $\alpha \in \mathbb{R}^{s \times p}$ and $\beta \in \mathbb{R}^{q \times t}$, a linear operator $\llbracket \alpha, \beta \rrbracket : \mathbb{R}^{p \times q} \rightarrow \mathbb{R}^{s \times t}$ by

$$\llbracket \alpha, \beta \rrbracket(X) := \alpha X \beta. \quad (106)$$

The generalization of this construct to matrices $\alpha_1, \dots, \alpha_r \in \mathbb{R}^{s \times p}$ and $\beta_1, \dots, \beta_r \in \mathbb{R}^{q \times t}$, with r an arbitrary positive integer, leads to a linear operator

$$\llbracket \alpha_1, \beta_1 \mid \dots \mid \alpha_r, \beta_r \rrbracket := \sum_{k=1}^r \llbracket \alpha_k, \beta_k \rrbracket, \quad (107)$$

where the matrix pairs are separated by “ \mid ”s. Of particular importance are self-adjoint linear operators on the Hilbert space $\mathbb{R}^{p \times q}$ of the form (107) where $\alpha_1, \dots, \alpha_r \in \mathbb{R}^{p \times p}$ and $\beta_1, \dots, \beta_r \in \mathbb{R}^{q \times q}$ are such that for any $k = 1, \dots, r$, the matrices α_k and β_k are either both symmetric or both antisymmetric. Such an operator (107) is referred to as a *self-adjoint operator of grade r* , with the self-adjointness understood in the sense of the Frobenius inner product on $\mathbb{R}^{p \times q}$, so that $\llbracket \alpha, \beta \rrbracket^\dagger = \llbracket \alpha^T, \beta^T \rrbracket$.

Lemma 7 [22] *If $\alpha \in \mathbb{R}^{p \times p}$ and $\beta \in \mathbb{R}^{q \times q}$ are both antisymmetric, then the spectrum of $\llbracket \alpha, \beta \rrbracket$ is symmetric about the origin. If α and β are both symmetric and positive (semi-) definite, then $\llbracket \alpha, \beta \rrbracket$ is positive (semi-) definite, respectively.*

Whilst the operator (106) with nonsingular α and β is straightforwardly invertible, so that $\llbracket \alpha, \beta \rrbracket^{-1} = \llbracket \alpha^{-1}, \beta^{-1} \rrbracket$, the inverse of the operator from (107) with $r > 1$, in general, can only be computed using the vectorization of matrices [10, 18] as

$$\llbracket \alpha_1, \beta_1 \mid \dots \mid \alpha_r, \beta_r \rrbracket^{-1}(Y) = \text{vec}^{-1}(\gamma^{-1} \text{vec}(Y)),$$

provided $\gamma := \sum_{k=1}^r \beta_k^T \otimes \alpha_k$ is nonsingular. Here, $\text{vec}(Y)$ is the vector obtained by writing the columns of a matrix Y one underneath the other.